Green’s Theorem, Gauss’s Theorem & Stoke’s Theorem

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Contents

Green's Theorem

If $\phi(x, y), \psi(x, y), \dfrac{\partial \phi}{\partial y}$ **and** $\dfrac{\partial \psi}{\partial x}$ **be continuous functions over a region R bounded by simple closed curve C in xy-plane, then**

$$\oint_C (\phi\, dx + \psi\, dy) = \iint_R \left(\frac{\partial \psi}{\partial x} - \frac{\partial \phi}{\partial y} \right) dx\, dy$$

Green's Theorem...

Green's Theorem in vector form:

$$\int_C \vec{F}.d\vec{r} = \iint_R \left(\nabla \times \vec{F}\right).\hat{k}\, dR$$

where $\vec{F} = \phi\,\hat{i} + \psi\,\hat{j}$, $\vec{r} = x\,\hat{i} + y\,\hat{j}$

$\hat{k}$ **is the unit vector along z-axis**

and $dR = dx\,dy$

Example

Q: A vector field

$\vec{F} = \sin y\,\hat{i} + x\left(1+\cos y\right)\hat{j}$

Evaluate the integral $\int_C \vec{F}.d\vec{r}$ **where C is the circular path given by** $x^2+y^2=a^2$**.**

Sol:

$$\int_C \vec{F}.d\vec{r} = \int_C \left[\sin y\,\hat{i} + x\left(1+\cos y\right)\hat{j}\right].\left(\hat{i}\,dx + \hat{j}\,dy\right)$$

$$= \int_C \sin y\,dx + x\left(1+\cos y\right)dy$$

Example...

On applying
Green's theorem, we have

$$\oint_C (\phi\, dx + \psi\, dy) = \iint_R \left(\frac{\partial \psi}{\partial x} - \frac{\partial \phi}{\partial y} \right) dx\, dy$$

$$= \iint_R \left[(1 + \cos y) - \cos y \right] dx\, dy$$

where R is the circular plane surface of radius a.

$$\oint_C (\phi\, dx + \psi\, dy) = \iint_R dx\, dy = area\ of\ circle$$

$$\Rightarrow \int_C \vec{F} . d\vec{r} = \pi a^2$$

Area of Plane Region by Green's Theorem

$$Put\ \psi = x\ and\ \phi = -y\ \ in\ Green's\ theorem$$

$$\int_C (x\,dy - y\,dx) = \iint_A \left[1 - (-1)\right] dx\,dy$$

$$= 2\iint_A dx\,dy = 2 \times Area$$

$$\Rightarrow Area = \frac{1}{2}\int_C (x\,dy - y\,dx)$$

Gauss's Divergence Theorem

The surface integral of the normal component of vector function $\vec{F}$ taken around a closed surface S is equal to the integral of divergence of $\vec{F}$ taken over the volume V enclosed by the surface S.

$$\iint_S \vec{F}.\hat{n}\, dS = \iiint_V div\, \vec{F}\, dV$$

Example

Q: Evaluate $\iint_S \vec{F}.\hat{n}\, dS$

where S is the surface of the sphere $x^2+y^2+z^2=16$ and $\vec{F} = 3x\,\hat{i} + 4y\,\hat{j} + 5z\,\hat{k}$

Sol: $\iint_S \vec{F}.\hat{n}\, dS = \iiint_V div\, \vec{F}\, dV$

$$div\, \vec{F} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right).\left(3x\,\hat{i} + 4y\,\hat{j} + 5z\,\hat{k}\right)$$

$$div\, \vec{F} = 3 + 4 + 5 = 12$$

Example...

$$\iint_S \vec{F}.\hat{n}\, dS = \iiint_V div\, \vec{F}\, dV = \iiint_V 14\, dV$$

$$= 12 \times V$$

$$= 12 \times \frac{4}{3}\pi\left(4\right)^3$$

$$\boxed{\iint_S \vec{F}.\hat{n}\, dS = 1024}$$

Example

Q: Evaluate $\iint_S \vec{F}.\hat{n}\, dS$
where S is the surface of the cube bounded by x=0, x=1, y=0, y=1, z=0, z=1 and $\vec{F} = 4xz\,\hat{i} - y^2\,\hat{j} + yz\,\hat{k}$

Sol: $\iint_S \vec{F}.\hat{n}\, dS = \iiint_V div\,\vec{F}\, dV$

$$div\,\vec{F} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right).\left(4xz\,\hat{i} - y^2\,\hat{j} + yz\,\hat{k}\right)$$

$$div\,\vec{F} = 4z - 2y + y = 4z - y$$

Example...

$$\iint_S \vec{F}.\hat{n}\, dS = \iiint_V div\, \vec{F}\, dV$$

$$= \iiint_V (4z - y)\, dx\, dy\, dz$$

$$= \int_0^1 \int_0^1 \int_0^1 (4z - y)\, dz\, dy\, dx$$

$$= \int_0^1 \int_0^1 \left[2z^2 - yz\right]_0^1 dy\, dx$$

Example...

$$\iint_S \vec{F}.\hat{n}\,dS = \int_0^1\int_0^1 (2-y)\,dy\,dx$$

$$= \int_0^1 \left[2y - \frac{y^2}{2}\right]_0^1 dx = \int_0^1 \frac{3}{2}dx$$

$$\iint_S \vec{F}.\hat{n}\,dS = \frac{3}{2}$$

Stoke's Theorem

Surface integral of the component of curl $\vec{F}$ along the normal to the surface S, taken over the surface S and bounded by curve C is equal to the line integral of the vector point function $\vec{F}$ taken along closed curve C.

$$\oint \vec{F}.d\vec{r} = \iint_S curl\,\vec{F}.\hat{n}\,dS$$

where n is unit external normal vector to surface dS.

Example

Q: Using Stoke's theorem evaluate

$$\int_C \left[\left(2x - y\right) dx - yz^2 dy - y^2 z\, dz\right]$$

where C is the circle x²+y²=1, corresponding to the surface of sphere of unit radius.

Sol:

$$\int_C \left[\left(2x - y\right) dx - yz^2 dy - y^2 z\, dz\right]$$

$$= \int_C \left[\left(2x - y\right)\hat{i} - yz^2\,\hat{j} - y^2 z\,\hat{k}\right].\left(dx\,\hat{i} - dy\,\hat{j} - dz\,\hat{k}\right)$$

Example...

By Stoke's theorem

$$\oint \vec{F}.d\vec{r} = \iint_s curl\,\vec{F}.\hat{n}\,dS$$

$$curl\,\vec{F} = \vec{\nabla} \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 2x - y & -yz^2 & -y^2 z \end{vmatrix}$$

$$= \left(-2yz + 2yz\right)\hat{i} - \left(0 - 0\right)\hat{j} + \left(0 + 1\right)\hat{k}$$

$$\Rightarrow curl\,\vec{F} = \hat{k}$$

Example...

$$\oint \vec{F}.d\vec{r} = \iint_S \hat{k}.\hat{n}\, dS$$

$$= \iint_S \hat{k}.\hat{n} \frac{dx\, dy}{\hat{n}.\hat{k}}$$

$$= \iint_S dx\, dy = Area\ of\ cirle = \pi$$

$$\Rightarrow \int_C \left[\left(2x - y\right) dx - yz^2 dy - y^2 z\, dz \right] = \pi$$

Exercise

Q1: Evaluate $\iint_S \vec{F}.\hat{n}\, dS$
over the entire surface of the region above xy-plane bounded by the cone $z^2=x^2+y^2$ and the plane z=4, if

$$\vec{F} = 4xz\,\hat{i} + xyz^2\,\hat{j} + 3z\,\hat{k}$$

Q2: Using Green's theorem, find the region in the first quadrant bounded by the curves y=x, y=1/x, y=x/4.

Exercise

Q3: Apply Stoke's theorem to evaluate $\int_C \vec{v}.\overrightarrow{dr}$, where $\vec{v} = y^2\hat{i} + xy\,\hat{j} + xz\,\hat{k}$ and C is the bounding curve of the hemisphere $x^2+y^2+z^2=9$, $z>0$, oriented in the positive direction.

Answer

Q1: 320π

Q2: log 2

Q3: 0

THANK YOU