Vector Integration

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Integration of Vectors

Let $\vec{R}(t) = R_1(t)\hat{i} + R_2(t)\hat{j} + R_3(t)\hat{k}$ **be a vector which depends on variable t and** $R_1(t), R_2(t), R_3(t)$ **are continuous in the given interval. Then**

$$\int \overrightarrow{R}(t)\,dt = \hat{i}\int R_1(t)\,dt + \hat{j}\int R_2(t)\,dt + \hat{k}\int R_3(t)\,dt$$

is called indefinite integral of $\vec{R}(t)$**.**

Integration of Vectors…

If $\vec{S}(t)$ be a vector such that

$$\vec{R}(t) = \frac{d}{dt}\vec{S}(t)$$

then

$$\int \vec{R}(t)\,dt = \int \frac{d}{dt}\vec{S}(t)\,dt$$

$$= \vec{S}(t) + C$$

Where C is a constant.

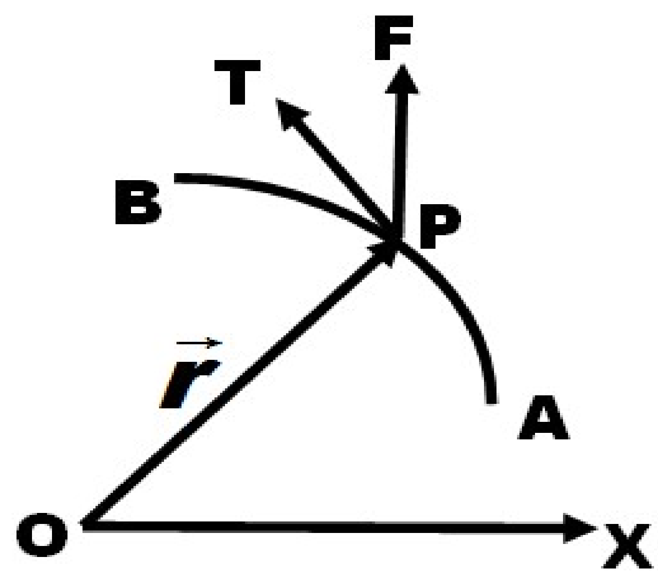
Integration of Vectors…

If integration is definite, then

$$\int_a^b \overrightarrow{R}(t)\,dt = \left[\vec{S}(t)\right]_a^b$$

$$= \vec{S}(b) - \vec{S}(a)$$

Line Integral



Let $\vec{F}(x, y, z)$ be a vector function and a curve AB.
Line integral of a vector function $\vec{F}$ along the curve AB is defined as integral of the component of $\vec{F}$ along the tangent to the curve AB.

Line Integral…

Component of $\vec{F}$ along a tangent PT at P = Dot product of $\vec{F}$ and unit vector along PT

$$= \vec{F}.\frac{\overrightarrow{dr}}{ds}$$

$\frac{\overrightarrow{dr}}{ds}$ is a unit vector along tangent PT

Line Integral = $\int_c \left(\vec{F}.\frac{\overrightarrow{dr}}{ds}\right) ds = \int_c \vec{F}.\overrightarrow{dr}$

Line Integral...

$$\text{Line Integral} = \int_c \left(\vec{F} \cdot \frac{\overrightarrow{dr}}{ds} \right) ds = \int_c \vec{F} \cdot \overrightarrow{dr}$$

Note: If the path of integration is a closed curve then notation of integration is $\oint$.

Example

Q: If a force

$$\vec{F} = 2x^2y\,\hat{i} + 3xy\,\hat{j}$$

displaces a particle in xy- plane from (0,0) to (1,4) along a curve $y = 4x^2$. Find the work done.

Sol: Work done = $\int_c \vec{F}.\overrightarrow{dr}$

$$\vec{r} = x\,\hat{i} + y\,\hat{j}$$

$$\overrightarrow{dr} = dx\,\hat{i} + dy\,\hat{j}$$

Example...

$$\text{Work done} = \int_c \vec{F}.\overrightarrow{dr}$$

$$= \int_c \left(2x^2 y\,\hat{i} + 3xy\,\hat{j}\right).\left(dx\,\hat{i} + dy\,\hat{j}\right)$$

$$= \int_c \left(2x^2 y\,dx + 3xy\,dy\right)$$

$$= \int_0^1 \left[2x^2\left(4x^2\right)dx + 3x\left(4x^2\right)8x\,dx\right]$$

$$= 104\int_0^1 x^4 dx = 104\left(\frac{x^5}{5}\right)_0^1 = \frac{104}{5}$$

Surface Integral

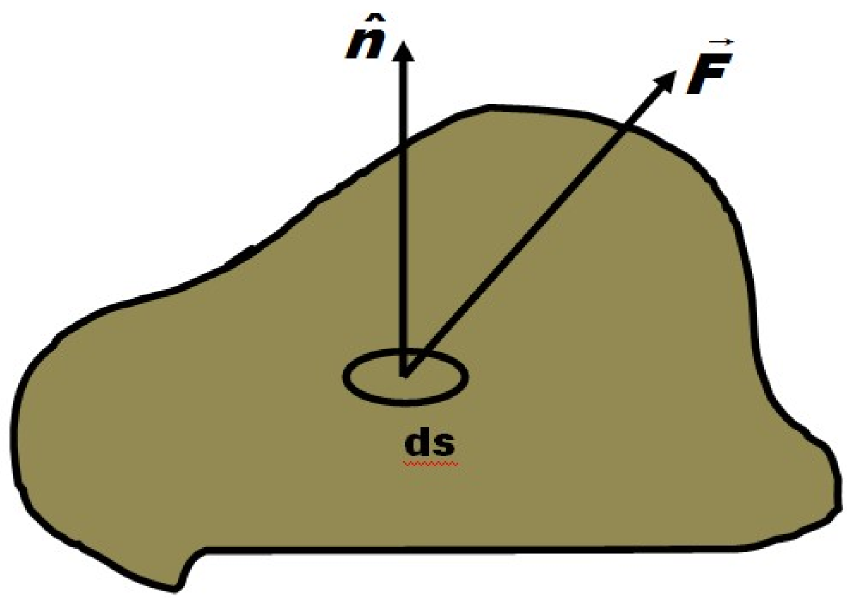
Let $\vec{F}$ be a vector function and S be the surface.

Surface integral of a vector function $\vec{F}$ over the surface S is defined as the integral of the components of $\vec{F}$ along the normal to the surface .

Surface Integral...

Component of $\vec{F}$ along normal $= \vec{F}.\hat{n}$

Where n is the unit normal vector to an element ds



$$\hat{n} = \frac{grad\ f}{\left|grad\ f\right|}$$

$$ds = \frac{dx\ dy}{\left(\hat{n}.\hat{k}\right)}$$

Surface Integral...

Surface integral of F over S $= \iint\limits_{s} \left(\vec{F}.\hat{n}\right) dS$

Note:

If $\iint\limits_{s} \left(\vec{F}.\hat{n}\right) dS = 0$ **then** $\vec{F}$ **is said to be solenoidal vector point function.**

Example

Q: Evaluate $\iint_s \left(\vec{F}.\hat{n}\right) ds$

where $\vec{F} = 18z\,\hat{i} - 12\,\hat{j} + 3y\,\hat{k}$ **and S is the part of the plane 2x+3y+6z=12 in the first octant.**

Sol: Here, $\vec{F} = 18z\,\hat{i} - 12\,\hat{j} + 3y\,\hat{k}$

S = 2x+3y+6z-12

Normal to S = $\nabla.S$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right).\left(2x + 3y + 6z - 12\right)$$

Example...

$$\text{Normal to } S = 2\hat{i} + 3\hat{j} + 6\hat{k}$$

$$\text{Unit normal vector} = \frac{2\hat{i} + 3\hat{j} + 6\hat{k}}{\sqrt{4 + 9 + 36}}$$

$$= \frac{2\hat{i} + 3\hat{j} + 6\hat{k}}{7}$$

$$\iint \vec{F}.\hat{n}\, ds =$$

$$\iint \left(18z\,\hat{i} - 12\,\hat{j} + 3y\,\hat{k}\right). \frac{\left(2\hat{i} + 3\hat{j} + 6\hat{k}\right)}{7} \frac{dx\,dy}{\hat{n}.\hat{k}}$$

Example...

$$\iint \vec{F}.\hat{n}\, ds = \iint (36z - 36 + 18y)\frac{1}{7}\frac{dx\,dy}{\frac{(2\hat{i} + 3\hat{j} + 6\hat{k})}{7}.\hat{k}}$$

$$= \iint (36z - 36 + 18y)\frac{1}{7}\frac{dx\,dy}{\frac{6}{7}}$$

2x+3y+6z=12

in xy-plane, z=0

2x+3y=12 $\Rightarrow y = \frac{12 - 2x}{3}$

Example...

$$\iint \vec{F}.\hat{n}\, ds = \iint \left[6\left(12 - 2x - 3y\right) - 36 + 18y\right]\frac{dx\,dy}{6}$$

$$= \frac{1}{6}\iint \left(72 - 12x - 18y - 36 + 18y\right)dx\,dy$$

$$= \iint \left(6 - 2x\right)dx\,dy$$

$$= \int_0^6 \int_0^{\frac{12-2x}{3}} \left(6 - 2x\right)dx\,dy$$

Example...

$$\iint \vec{F}.\hat{n}\,ds = \int_0^6 \left[6y - 2xy\right]_0^{\frac{12-2x}{3}} dx$$

$$= \int_0^6 \left[6\left(\frac{12-2x}{3}\right) - 2x\left(\frac{12-2x}{3}\right)\right] dx$$

$$= \frac{1}{3}\int_0^6 \left(72 - 12x - 24x + 4x^2\right) dx$$

$$= \frac{1}{3}\left[72x - 36\frac{x^2}{2} + 4\frac{x^3}{3}\right]_0^6$$

Example...

$$\iint \vec{F}.\hat{n}\,ds = \frac{1}{3}\left[72(6) - 36\frac{6^2}{2} + 4\frac{6^3}{3}\right]$$

$$= \frac{1}{3}\left[432 - 648 + 288\right] \quad = \frac{72}{3}$$

$$\iint \vec{F}.\hat{n}\,ds = 24$$

Volume Integral

Let $\vec{F}$ be a vector function and volume V enclosed by a closed surface.

The volume integral = $\iiint_V \vec{F}\, dV$

Example

Q: If $\vec{F} = 2z\,\hat{i} - x\,\hat{j} + y\,\hat{k}$

Evaluate $\iiint_V \vec{F}\,dV$

where, V is the region bounded by the surfaces x=0, y=0, x=2, y=4, $z=x^2$, z=2.

Sol: $\iiint_V \vec{F}\,dV = \iiint \left(2z\,\hat{i} - x\,\hat{j} + y\,\hat{k}\right) dx\,dy\,dz$

$$= \int_0^2 dx \int_0^4 dy \int_{x^2}^2 \left(2z\,\hat{i} - x\,\hat{j} + y\,\hat{k}\right) dz$$

Example...

$$\iiint_V \vec{F}\, dV = \int_0^2 dx \int_0^4 dy \left[z^2\,\hat{i} - xz\,\hat{j} + yz\,\hat{k} \right]_{x^2}^{2}$$

$$= \int_0^2 dx \int_0^4 \left(4\,\hat{i} - 2x\,\hat{j} + 2y\,\hat{k} - x^4\,\hat{i} + x^3\,\hat{j} - x^2 y\,\hat{k} \right) dy$$

$$= \int_0^2 \left[4y\hat{i} - 2xy\hat{j} + y^2\hat{k} - x^4 y\hat{i} + x^3 y\hat{j} - x^2 \frac{y^2}{2}\hat{k} \right]_0^4 dx$$

$$= \int_0^2 \left(16\hat{i} - 8x\hat{j} + 16\hat{k} - 4x^4\hat{i} + 4x^3\hat{j} - 8x^2\hat{k} \right) dx$$

Example...

$$\iiint_V \vec{F}\, dV = \int_0^2 \left(16\hat{i} - 8x\hat{j} + 16\hat{k} - 4x^4\hat{i} + 4x^3\hat{j} - 8x^2\hat{k}\right) dx$$

$$= \left[16x\hat{i} - 4x^2\hat{j} + 16x\hat{k} - 4\frac{x^5}{5}\hat{i} + x^4\hat{j} - 8\frac{x^3}{3}\hat{k}\right]_0^2$$

$$= 32\hat{i} - 16\hat{j} + 32\hat{k} - \frac{128}{5}\hat{i} + 16\hat{j} - \frac{64}{3}\hat{k}$$

$$\boxed{\iiint_V \vec{F}\, dV = \frac{32}{5}\hat{i} + \frac{32}{3}\hat{k}}$$

Exercise

Q1: Find the work done in moving a particle in the force field:

$$\vec{F} = 3x^2\hat{i} + \left(2xz - y\right)\hat{j} + z\,\hat{k}$$

along the curve $x^2=4y$ and $3x^3=8z$ from $x=0$ to $x=2$.

Q2: Evaluate $\iint_s \left(yz\,\hat{i} + xz\,\hat{j} + xy\,\hat{k}\right).\overrightarrow{ds}$ **where S is the surface of the sphere $x^2+y^2+z^2=a^2$ in the first octant.**

Exercise

Q3: If $\vec{F} = \left(2x^2 - 3z\right)\hat{i} - 2xy\,\hat{j} - 4x\,\hat{k}$

Evaluate $\iiint_V \vec{F}\, dV$

where, V is the region bounded by the plane x=0, y=0, z=0 and 2x+2y+z=4

Answers

Q1: 16

Q2: $3a^4/8$

Q3: 8/3

THANK YOU